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# Entropy of the Binomial Field Interacting with the Two Entangled Atoms

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**Abstract** In this paper, we investigate entropy properties of the single-mode binomial field interacting with the two entangled two-level atoms. It is found that the different initial conditions lead to the different evolutions of entropy of the binomial field.

**Keywords** Entropy · Binomial field · Bell state · Entanglement

## 1 Introduction

Since the entropy theory about the interaction of the field with the atom was presented by Phoenix and Knight (PK) and co-workers [1–4], much attention has been focused on the properties of field (atom) entropy [5–14]. Because the PK entropy theory tells us that entropy is a very useful operational measure of the purity of the quantum mechanical state and the degree of the entanglement of the atom-field.

The main attention in the previous articles has been devoted to a squeezed state or a coherent state for a field and a disentangled state for atoms. Less attention has been paid to a binomial state interacting with entangled atoms. In this paper, we will study the entropy of the single-mode binomial field interacting with the two entangled two-level atoms.

This paper is organized as follows. Section 2 gives the solution of the model. Section 3 studies the entropy formalism of the binomial field. Section 4 investigates the numerical results of the field entropy. A conclusion is presented in Sect. 5.

## 2 The Model and Its Solution

The effective Hamiltonian of the model under consideration in this paper in the rotating-wave approximation can be written as

$$H = H_0 + V \quad (\hbar = 1), \tag{1}$$

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where

$$H_0 = \omega a^+ a + \omega_0(\sigma_z^{(1)} + \sigma_z^{(2)}), \tag{2}$$

$$V = g(a^+\sigma_-^{(1)} + a\sigma_+^{(1)} + a^+\sigma_-^{(2)} + a\sigma_+^{(2)}), \tag{3}$$

where $a^+(a)$ is creation (annihilation) operator of the single-mode binomial field, $\sigma_z^{(1)}$, $\sigma_z^{(2)}$, $\sigma_-^{(1)}$, $\sigma_+^{(1)}$, $\sigma_-^{(2)}$, and $\sigma_+^{(2)}$ are the usual pseudospin operators acting in the space of atomic states and obey the commutation relations $[\sigma_+^{(1)}, \sigma_-^{(1)}] = \sigma_z^{(1)}$, $[\sigma_z^{(1)}, \sigma_\pm^{(1)}] = \pm\sigma_\pm^{(1)}$, $[\sigma_+^{(2)}, \sigma_-^{(2)}] = \sigma_z^{(2)}$, and $[\sigma_z^{(2)}, \sigma_\pm^{(2)}] = \pm\sigma_\pm^{(2)}$, $\omega$ and $\omega_0$ are the frequencies of the single-mode binomial field and the atomic transition, respectively, $g$ is the atoms-field coupling coefficient.

We consider that at $t = 0$ the two atoms are in one of the following Bell states which are also regarded as the maximal-entangled states

$$|\varphi_A(0)\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + i|ee\rangle), \tag{4}$$

$$|\phi_A(0)\rangle = \frac{1}{\sqrt{2}}(|ge\rangle + i|eg\rangle). \tag{5}$$

The two Bell states have been generated in a thermal cavity [15]. Unified atomic initial state can be written as

$$|\psi_A(0)\rangle = \gamma_1|ee\rangle + \gamma_2|eg\rangle + \gamma_3|ge\rangle + \gamma_4|gg\rangle. \tag{6}$$

The above two Bell states can be obtained by choose appropriate values of $\gamma_1$, $\gamma_2$, $\gamma_3$, $\gamma_4$ in (6).

The light field is initially in the binomial state

$$|\psi_F(0)\rangle = |p, M\rangle = \sum_{n=0}^{M} B_n^M |n\rangle, \tag{7}$$

where

$$B_n^M = \left[\frac{M!}{n!(M-n)!} p^n (1-p)^{M-n}\right]^{1/2}, \tag{8}$$

where $M$ is the maximum photon number present in the field and $p$ is the characteristic probability of having each photon occurring. The single-mode binomial state of the quantized electromagnetic field was presented by Stoler et al. in 1985 [16]. From (7) and (8) we can found that given any (finite) $M$, if $p = 0$, $|p, M\rangle$ is reduced to the vacuum state $|0\rangle$. On the other hand, if $p = 1$, we obtain the number state $|n = M\rangle$. Moreover, in the limit $p \to 0$ and $M \to \infty$, but with $pM = \alpha^2$ constant, the binomial distribution turns into a Poissonian distribution and $|p, M\rangle$ becomes a coherent state $|\alpha\rangle$. It is interesting to found that by changing two parameters ($p$ and $M$) in a binomial state, we can obtain fundamentally different states of the electromagnetic field.

The initial state of the system is a decoupled pure state, and the state vector can be written as

$$\begin{aligned}|\psi_{FA}(0)\rangle &= |\psi_F(0)\rangle \otimes |\psi_A(0)\rangle \\ &= \sum_{n=0}^{M} B_n^M (\gamma_1|ee,n\rangle + \gamma_2|eg,n\rangle + \gamma_3|ge,n\rangle + \gamma_4|gg,n\rangle). \end{aligned} \tag{9}$$

As the time goes, the evolution of the system in the interaction picture is governed by the state vector

$$|\psi_{FA}(t)\rangle = |a\rangle|ee\rangle + |b\rangle|eg\rangle + |c\rangle|ge\rangle + |d\rangle|gg\rangle, \tag{10}$$

where

$$|a\rangle = \sum_{n=0}^{M} a(n,t)|n\rangle, \tag{11}$$

$$|b\rangle = \sum_{n=0}^{M} b(n,t)|n\rangle, \tag{12}$$

$$|c\rangle = \sum_{n=0}^{M} c(n,t)|n\rangle, \tag{13}$$

$$|d\rangle = \sum_{n=0}^{M} d(n,t)|n\rangle. \tag{14}$$

Solving the Schrodinger equation

$$i\frac{\partial}{\partial t}|\psi_{FA}(t)\rangle = V|\psi_{FA}(t)\rangle, \tag{15}$$

we can obtain

$$\begin{aligned} a(n,t) = \varepsilon_3(n+1) - i\sqrt{\frac{n+1}{2(2n+3)}}[\varepsilon_1(n+1)\sin(\sqrt{2(2n+3)}gt) \\ - \varepsilon_2(n+1)\cos(\sqrt{2(2n+3)}gt)] \quad (n = 0,\ 1,\ 2,\ldots,M), \end{aligned} \tag{16}$$

$$\begin{aligned} b(n,t) = c(n,t) = \frac{1}{2}\varepsilon_1(n)\cos(\sqrt{2(2n+1)}gt) \\ + \frac{1}{2}\varepsilon_2(n)\sin(\sqrt{2(2n+1)}gt) \quad (n = 1,\ 2,\ 3,\ldots,M), \end{aligned} \tag{17}$$

$$\begin{aligned} d(n,t) = \varepsilon_4(n-1) - i\sqrt{\frac{n}{2(2n-1)}}[\varepsilon_1(n-1)\sin(\sqrt{2(2n-1)}gt) \\ - \varepsilon_2(n-1)\cos(\sqrt{2(2n-1)}gt)] \quad (n = 2,\ 3,\ 4,\ldots,M), \end{aligned} \tag{18}$$

$$b(0,t) = c(0,t) = \frac{1}{2}(\gamma_2+\gamma_3)B_0^M\cos(\sqrt{2}gt) - i\frac{1}{\sqrt{2}}\gamma_4 B_1^M\sin(\sqrt{2}gt), \tag{19}$$

$$d(0,t) = \gamma_4 B_0^M, \tag{20}$$

$$d(1,t) = \gamma_4 B_1^M\cos(\sqrt{2}gt) - i\frac{1}{\sqrt{2}}(\gamma_2+\gamma_3)B_0^M\sin(\sqrt{2}gt), \tag{21}$$

where

$$\varepsilon_1(n) = (\gamma_2 + \gamma_3)B_n^M, \tag{22}$$

$$\varepsilon_2(n) = -i\sqrt{\frac{2}{2n+1}}(\gamma_1\sqrt{n}B_{n-1}^M + \gamma_4\sqrt{n+1}B_{n+1}^M), \tag{23}$$

$$\varepsilon_3(n) = \frac{1}{2n+1}(\gamma_1(n+1)B_{n-1}^M - \gamma_4\sqrt{n(n+1)}B_{n+1}^M), \tag{24}$$

$$\varepsilon_4(n) = \frac{1}{2n+1}(\gamma_4 n B_{n+1}^M - \gamma_1\sqrt{n(n+1)}B_{n-1}^M). \tag{25}$$

## 3 Evolution Formalism of the Field Entropy

Since we have assumed that the two two-level atoms and the single-mode binomial field are initially in a disentangled pure state, the total entropy of the system is equal to zero. In terms of the triangle inequality of the entropy [17]

$$|S_A(t) - S_F(t)| \le S_{AF}(t) \le |S_A(t) + S_F(t)|, \tag{26}$$

we can find that the reduced entropies of the two subsystems are identical, namely $S_A(t) = S_F(t)$. Hence, the field entropy can be obtained by operating the atomic entropy.

The reduced density matrix of the atoms is given by

$$\rho_A(t) = \mathrm{Tr}_F|\psi_{FA}(t)\rangle\langle\psi_{FA}(t)| = \begin{bmatrix} \langle a|a\rangle & \langle b|a\rangle & \langle c|a\rangle & \langle d|a\rangle \\ \langle a|b\rangle & \langle b|b\rangle & \langle c|b\rangle & \langle d|b\rangle \\ \langle a|c\rangle & \langle b|c\rangle & \langle c|c\rangle & \langle d|c\rangle \\ \langle a|d\rangle & \langle b|d\rangle & \langle c|d\rangle & \langle d|d\rangle \end{bmatrix}. \tag{27}$$

The field entropy can be defined as follows [18]

$$S_F(t) = S_A(t) = -\mathrm{Tr}_A[\rho_A(t)\ln\rho_A(t)] = -\sum_{j=1}^{4}\lambda_A^j \ln\lambda_A^j, \tag{28}$$

where $\lambda_A^j$ $(j = 1, 2, 3, 4)$ is the eigenvalue of reduced density matrix of the atoms.

The behavior of the field (atom) entropy reflects the behavior of the entanglement between the atoms and the field. The higher the entropy, the greater the entanglement. If the field (atom) subsystem is in a pure state, the entropy of this subsystem is equal to zero. On the other hand, if the entropy is larger than zero, the field (atom) subsystem is in a statistical mixture state and the two subsystems are entangled.

## 4 Numerical Results and Discussions

On the basis of the analytical solution presented in the previous sections, we will discuss the numerical results of the field entropy $S_F(t)$ given by (28). For the atoms initially in different Bell state and the field initially in different binomial state, the numerical results of $S_F(t)$ are investigated in detail. Figures 1, 2 and 3 show the effect of the initial field on the evolution of the field entropy for the atoms initially in Bell state $|\varphi_A(0)\rangle = \frac{1}{\sqrt{2}}(|gg\rangle + i|ee\rangle)$.

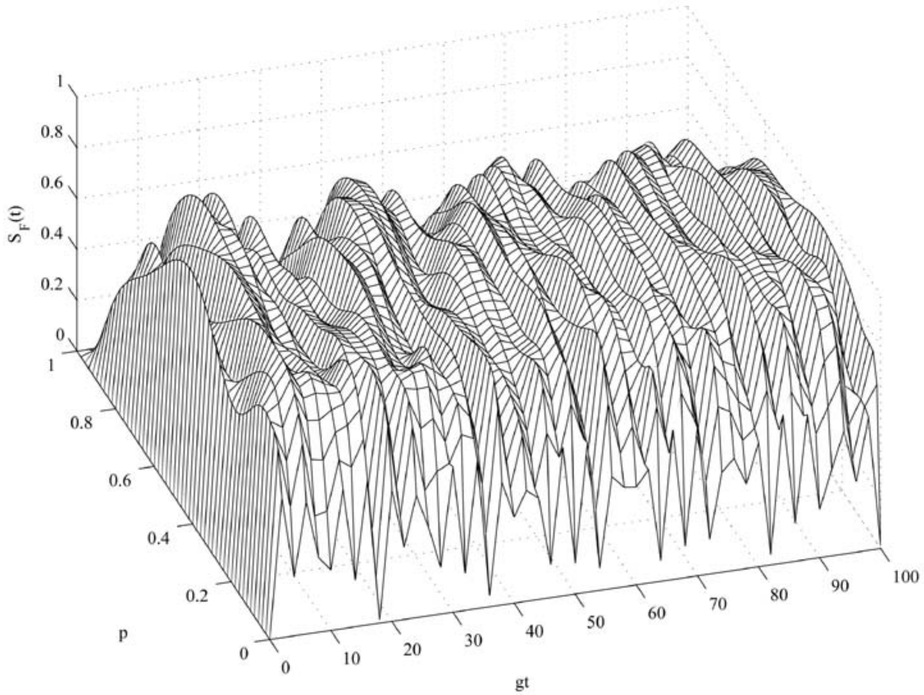


**Fig. 1** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 20$, the atoms initially in Bell state $|\varphi_A(0)\rangle$

In these three figures, we plotted the field entropy $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the different maximum photon number $M$. Figures 4, 5 and 6 display the similar numerical results for the atoms initially in Bell state $|\phi_A(0)\rangle = \frac{1}{\sqrt{2}}(|ge\rangle + i|eg\rangle)$.

By the great number of numerical computations, it is found that the numerical result of $S_F(t)$ for the atoms initially in Bell state $|\varphi_A(0)\rangle$, comparing Figs. 1, 2 and 3 with Figs. 4, 5 and 6, is similar to the numerical results of $S_F(t)$ for the atoms initially in Bell state $|\phi_A(0)\rangle$ regardless of the initial state of the binomial field. But the value of the field entropy for the atoms initially in Bell state $|\varphi_A(0)\rangle$ is larger than the value of the field entropy for the atoms initially in Bell state $|\phi_A(0)\rangle$. This result corresponds with the fact that the degree of the entanglement between the binomial field and the atoms initially in Bell state $|\varphi_A(0)\rangle$ is larger than the degree of the entanglement between the binomial field and the atoms initially in Bell state $|\phi_A(0)\rangle$.

On the other hand, when $M$ is very small, the field entropy exhibits irregular oscillation regardless of the value of $p$ (see Figs. 1 and 4). When $M$ is large enough, the field entropy still exhibits irregular oscillation for the small values of $p$, but the field entropy exhibits regular oscillation for the large values of $p$ (see Figs. 3 and 6). The other important fact is that the field entropy is nearly equal to zero, regardless of the value of $M$, for the very large values of $p$ ($p$ next to 1). In the other words, the binomial field and the atoms are nearly disentangled for the large values of $p$.

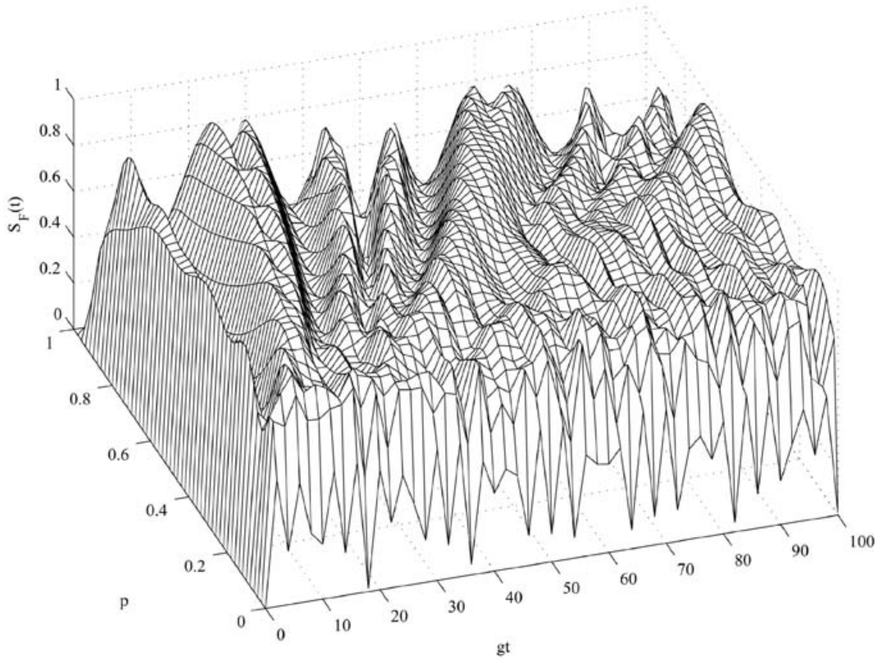


**Fig. 2** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 60$, the atoms initially in Bell state $|\varphi_A(0)\rangle$

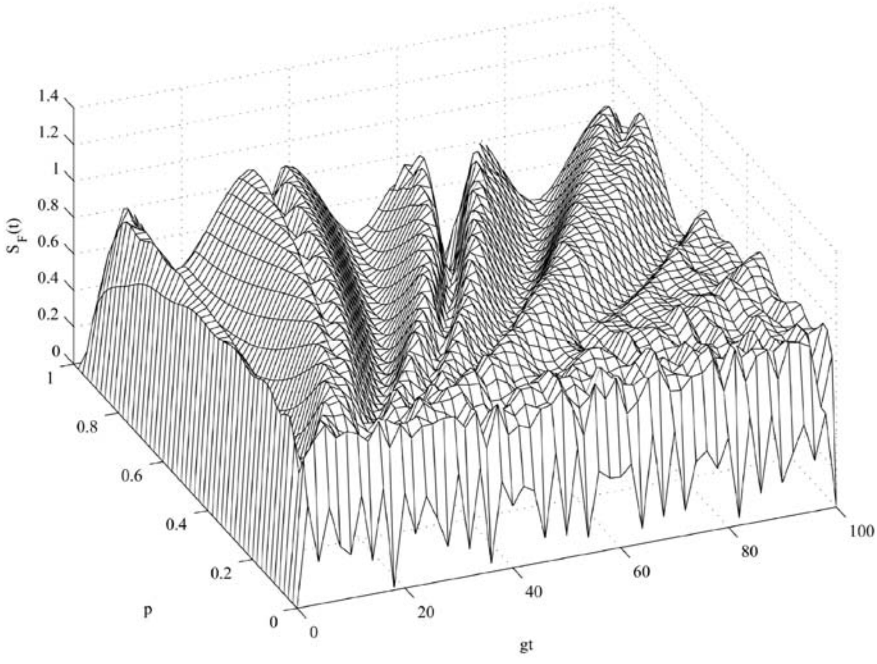


**Fig. 3** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 120$, the atoms initially in Bell state $|\varphi_A(0)\rangle$

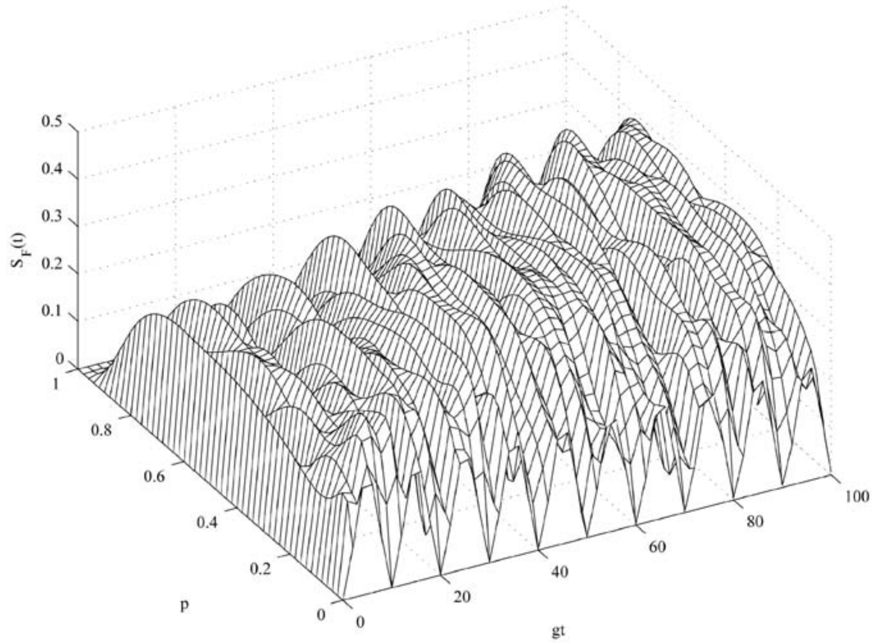


**Fig. 4** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 20$, the atoms initially in Bell state $|\phi_A(0)\rangle$

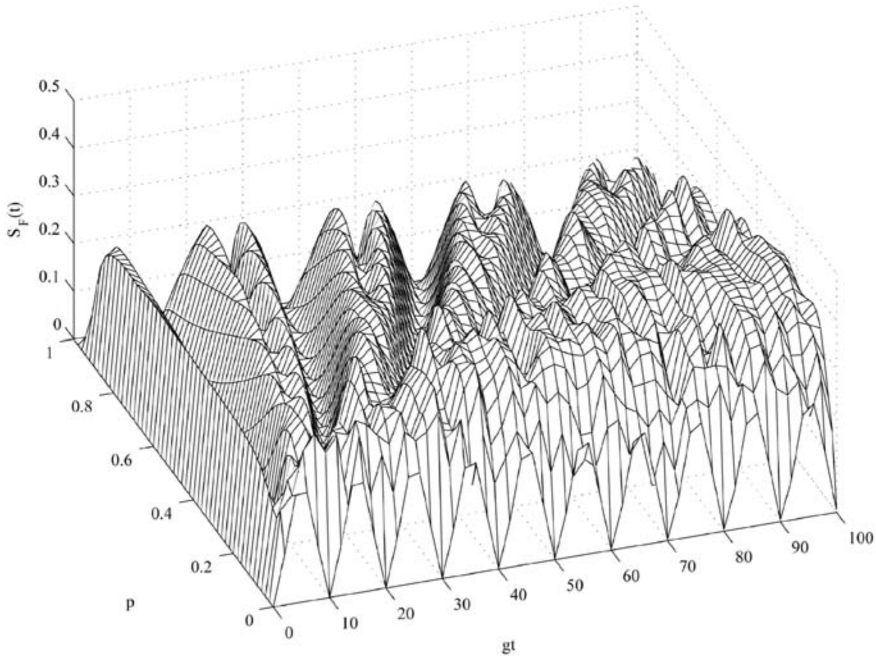


**Fig. 5** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 60$, the atoms initially in Bell state $|\phi_A(0)\rangle$

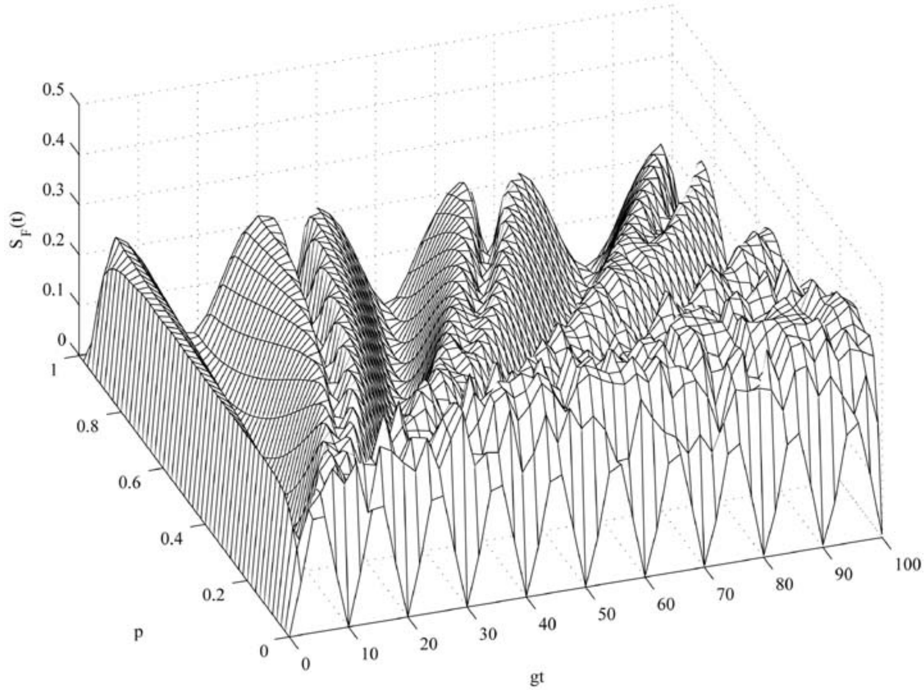


**Fig. 6** Diagram of $S_F(t)$ changing with the time $gt$ and the characteristic probability $p$ for the maximum photon number $M = 120$, the atoms initially in Bell state $|\phi_A(0)\rangle$

## 5 Conclusions

In this paper, we have studied the evolution of the binomial field interacting with the two two-level atoms initially in the different Bell state and examined the influence of the initial conditions on the evolution of the field entropy. It is found that the field entropy has the similar evolution for the different initial state of the two atoms except one situation in which the different initial states of the atoms lead to the different values of the field entropy (the degree of the entanglement between the binomial field and the two atoms). On the other hand, we can get the different evolution properties of the field entropy by controlling $M$ and $p$, that is, the parameters of the binomial field have an important effect on the evolution properties of the field entropy. Our numeral results may be useful for the relative experiments.

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